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On a Quarter-Symmetric Metric Connection in an (ε)-Kenmotsu Manifold

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Abstract: The object of the present paper is to study a quarter-symmetric metric connection in an (ε)-Kenmotsu manifold. We study some curvature properties of an (ε)-Kenmotsu manifold with respect to the quarter-symmetric metric connection.

Keywords: quarter-symmetric metric connection, (ε)-Kenmotsu manifold, locally ϕ-symmetric, ϕ-recurrent.
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1. Introduction

A semi-symmetric linear connection on a differentiable manifold was first introduced by Friedmann and Schouten[1] in 1924. Hayden[2] introduced and studied a semi-symmetric metric connection on a Riemannian manifold. Duggal and Sharma[3] studied a semi-symmetric metric connection on a semi-Riemannian manifold. The quarter-symmetric connection generalizes the semi-symmetric connection. The semi-symmetric metric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed point is metric and semi-symmetric[4]. In 1975, Golab[5] introduced and studied quarter symmetric connection in a differentiable manifold. A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection[5] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X, Y)=\tilde{\nabla}_X Y-\tilde{\nabla}_Y X-[X, Y]$$

is of the form

$$\tilde{T}(X, Y)=\eta(Y)\phi X-\eta(X)\phi Y. \quad (1.1)$$

where η is 1-form and ϕ is a tensor of type (1, 1). In addition, a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition[6]

$$(\tilde{\nabla}_X g)(Y, Z)=0, \quad (1.2)$$

for all $X, Y, Z \in TM^n$, where TM^n is the Lie-algebra of vector fields of the manifold M^n and g be the Riemannian metric, then $\tilde{\nabla}$ is said to be quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$, then the quarter symmetric connection reduces to a semi-symmetric connection[5].

After Golab[5], Rastogi[7] continued the symmetric study of quarter-symmetric metric connection. In 1980, Mishra and Pandey[8] studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai[9] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifolds, Quarter-symmetric connection are also studied by Biswas and De[10], Singh[11], De and Mondal[12], De and De[13], Singh and Pandey[6] and many other.

On the other hand, the study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1993, Bejancu and Duggal[14] introduced the concept of (ε)-Sasakian manifolds and Xupeng and Xiaoli[15] established that these manifolds are real hyper-surfaces of indefinite Kaehlerian manifolds. De and Sarkar[16] introduced (ε)-Kenmotsu manifolds and studied various properties of (ε)-Kenmotsu manifold. A semi-symmetric metric connection in an (ε)-Kenmotsu manifold whose projective curvature tensor satisfies certain curvature conditions have been studied by Singh, Pandey, Pandey and Tiwari[17], Motivated by these studies, in this paper, we study some curvature properties of an (ε)-Kenmotsu manifold with respect to quarter-symmetric metric connection. The present paper is organized as follows:

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After preliminaries in section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and established relations between curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection in an (ε)-Kenmotsu manifold. Section 4 deals with quasi-projectively flat (ε)-Kenmotsu manifold with respect to quarter-symmetry metric connection. Section 5 is devoted to study ϕ-protectively flat (ε)-Kenmotsu manifold with respect to a quarter-symmetric connection. In the last section, we study (ε)-Kenmotsu manifold with respect to a quarter-symmetric connection satisfying $\tilde{P}.\tilde{S}=0$.

2. (ε)-Kenmotsu Manifold

An n-dimensional smooth manifold (M^n,g) is called an (ε)-almost contact metric manifold if [16,17]

(2.1) $\phi^2 X=-X+\eta(X)\xi$,

(2.2) $\eta(\xi)=1$,

(2.3) $g(\xi,\xi)=\varepsilon$,

(2.4) $\eta(X)=\varepsilon g(X,\xi)$,

(2.5) $g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y)$,

where ε is 1 or -1 according as ξ is space-like or time-like and rank ϕ is $n-1$.

It is important to mention that in the above definition ξ is never a light-like vector field. If

(2.6) $d\eta(X,Y)=g(X,\phi Y)$,

for every $X,Y\in TM^n$, then we say that M^n is an (ε)-contact metric manifold[17]. Also,

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(2.7) $\phi\xi = 0, \quad \eta o\phi = 0$.

If an (ε)-contact metric manifold satisfies

(2.8) $(\nabla_X\phi)(Y) = -g(X,\phi Y)\xi - \varepsilon\eta(Y)\phi X,$

where ∇ denotes the Riemannian connection of g, then M^n Is called an (ε)-Kenmotsu manifold[16].

An (ε)-almost contact metric manifold is an (ε)-Kenmotsu manifold[16] if and only if

(2.9) $\nabla_X\xi = \varepsilon(X - \eta(X)\xi).$

In an (ε)-Kenmotsu manifold, the following relations hold[16]

(2.10) $(\nabla_X\eta)(Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y),$

(2.11) $R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$

(2.12) $R(\xi,X)Y = \eta(Y)X - \varepsilon g(X,Y)\xi,$

(2.13) $R(X,Y)\phi Z = \phi R(X,Y)Z + \varepsilon\{g(Y,Z)\phi X - g(X,Z)\phi Y$

$+g(x,\phi Z)Y - g(Y,\phi Z)X\}.$

(2.14) $\eta(R(X,y)Z) = \varepsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)],$

(2.15) $S(X,\xi) = -(n-1)\eta(X),$

(2.16) $S(\phi X,\phi Y) = S(X,Y) + \varepsilon(n-1)\eta(X)\eta(Y).$

Definition 2.1. *An (ε)-Kenmotsu manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form*[17]

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(2.17) $S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)$,

where a and b are scalar functions of ξ. De and Sarkar[16] have given an example of (ε)-Kenmotsu manifold

Example: We consider the three dimensional manifold $M^3=\{(X,Y,Z)\in R^3$ $Z\neq 0\}$, where (X,y,Z) are the standard co-ordinates in R^3. The vector fields $e_1=Z\frac{\partial}{\partial X}$, $e_2=Z\frac{\partial}{\partial X}$, $e_3=-Z\frac{\partial}{\partial Z}$ are linearly independent at each point of the manifold. Let g be the Riemannian metric defined by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0$$

and

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=\varepsilon,$$

where $\varepsilon=\pm 1$. Let η be the 1-form defined by

$$\eta(Z)=\varepsilon g(Z,e_3) \text{ for any } Z\in TM^n.$$

Let ϕ be the (1, 1) tensor field defined by

$$\phi(e_1)=e_2, \phi(e_2)=e_1, \phi(e_3)=0.$$

Then using the linearity property of ϕ and g, we have

$$\eta(e_3)=1,\ \phi^2 Z=-Z+\eta(Z)e_3,$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W) \text{ for any } Z,W\in TM^n.$$

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$[e_1,e_2]=0, [e_1,e_3]=\varepsilon e_1, [e_2,e_3]=\varepsilon e_2.$$

The Riemannian connection ∇ of the metric g is given by

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